Vector-Valued Lagrangian Function in Multiobjective Optimization Problems

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1. Introduction

In this paper we shall consider the vector minimization problem with nondifferentiable functions. In [2], for such a problem, we gave necessary and sufficient conditions for Pareto optimality and weak Pareto optimality in terms of the saddle point of the scalarized Lagrangian. In this paper we shall define a vector-valued Lagrangian and its saddle point, and then give necessary and sufficient conditions for optimality, Pareto optimality, and weak Pareto optimality in terms of the saddle point of the vector-valued Lagrangian.

In § 2, we shall formulate the vector minimization problem and give some definitions and fundamental results, which play an important role in the following sections. In § 3, associated with the problem formulated in § 2, we shall define a vector-valued Lagrangian and three kinds of saddle points, and investigate their properties. In § 4, we shall give necessary and sufficient conditions for optimality, Pareto optimality and weak Pareto optimality in terms of the saddle points of the vector-valued Lagrangian.

Before going further, let us introduce the following notations. Let R^n be the n-dimensional Euclidean space and let R^n_+ be the nonnegative orthant of R^n. Let $x=(x_1, x_2,\dots, x_n)$ and $y=(y_1, y_2,\dots, y_n)$ be vectors in R^n. Then,

(i) $x \geqq y$ iff $x_i > y_i$, $i=1, 2,\dots, n$,

(ii) $x > y$ iff $x_i > y_i$, $i=1, 2,\dots, n$,

(iii) $x \geq y$ iff $x \geqq y$ and $x \neq y$.

If it is not the case that $x \geq y$ $(x > y)$, we write $x \ngeq y$ $(x \not> y)$.

The inner product of two vectors x and y in R^n will be denoted by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

All vectors are to be regarded as column vectors for purposes of matrix multiplication. For an $l \times m$ matrix A, A^T will denote the transposition of A.

2. Formulation of Problem and Fundamental Results

In this section, we shall formulate the vector minimization problem and give some definitions and fundamental results.

Let $f_1, f_2, \ldots, f_l, g_1, g_2, \ldots, g_m$ be real-valued functions defined on R^n, and let Q_0 be a non-empty subset of R^n. Then consider the following vector minimization problem:

$$\text{(P)} \begin{cases} \text{minimize} \quad f(x) \\ \text{subject to } x \in Q, \end{cases}$$

where $f(x) = (f_1(x), \ldots, f_l(x))$, $Q = \bigcap_{i=0}^{m} Q_i$, and $Q_i = \{x \in R^n \mid g_i(x) \leqq 0\}$, $i = 1, 2, \ldots, m$.

The following three kinds of solutions are well known as solutions to the problem (P).

DEFINITION 2. 1. A point $x^* \in Q$ is said to be an optimal solution to the problem (P) if it holds that

$$f(x^*) \leqq f(x) \qquad \text{for all } x \in Q.$$

DEFINITION 2. 2. A point $x^* \in Q$ is said to be a Pareto optimal solution to the problem (P) if there exists no $x \in Q$ such that

$$f(x) \leq f(x^*).$$

DEFINITION 2. 3. A point $x^* \in Q$ is said to be a weak Pareto optimal solution to the problem (P) if there exists no $x \in Q$ such that

$$f(x) < f(x^*).$$

We shall give two lemmas which play an important role in the following.

The following lemma gives necessary and sufficient conditions for weak Pareto optimality to the problem (P).

LEMMA 2. 1. *Let $f_1, f_2, \dots, f_l, g_1, g_2, \dots, g_m$ be convex functions defined on R^n, and let Q_0 be a convex set of R^n. Suppose that Slater's constraint qualification holds : there exists an $x^0 \in Q_0$ such that*

$$g(x^0) < 0,$$

where $g(x) = (g_1(x), g_2(x), \dots, g_m(x))$.

Then $x^* \in Q_0$ is a weak Pareto optimal solution to the problem (P) *if and only if there exist vectors $v^* \in R^l$ and $u^* \in R^m$ such that*

$$v^* \geq 0, \ u^* \geqq 0, \tag{2. 1}$$

$$v^* \cdot f(x^*) + u \cdot g(x^*) \leqq v^* \cdot f(x^*) + u^* \cdot g(x^*) \leqq v^* \cdot f(x) + u^* \cdot g(x), \quad {}^\forall x \in Q_0, {}^\forall u \in R^m_+, \tag{2. 2}$$

$$g(x^*) \leqq 0, \tag{2. 3}$$

$$u^* \cdot g(x^*) = 0. \tag{2. 4}$$

PROOF. Omitted. See Maeda [2], p. 115.

The following lemma gives necessary and sufficient conditions for Pareto optimality to the problem (P).

LEMMA 2. 2. *In addition to the hypotheses of Lemma* 2. 1, *suppose that the following regularity condition holds at $x^* \in Q$: for each i, there exists an $x^i \in Q$ such that*

$$f_j(x^i) < f_j(x^*), \qquad j = 1, 2, \dots, m, \ j \neq i.$$

Then $x^* \in Q_0$ is a Pareto optimal solution to the problem (P) *if and only if there exist vectors $v^* \in R^l$ and $u^* \in R^m$ such that*

$$v^* > 0, \ u^* \geqq 0, \tag{2. 5}$$

$$v^* \cdot f(x^*) + u \cdot g(x^*) \leqq v^* \cdot f(x^*) + u^* \cdot g(x^*) \leqq v^* \cdot f(x) + u^* \cdot g(x), \quad {}^\forall x \in Q_0, {}^\forall u \in R^m_+, \tag{2. 6}$$

$$g(x^*) \leqq 0, \tag{2. 7}$$

$$u^* \cdot g(x^*) = 0. \tag{2. 8}$$

PROOF. Omitted. See Maeda [2], pp. 116-117.

3. Vector-Valued Lagrangian and Its Saddle Point

In this section, associated with the problem (P), we shall define a vector-valued Lagrangian and three kinds of saddle points, and investigate their properties.

Let $\mathcal{U}$ be the set of all nonnegative $l \times m$ matrices. Then, associated with the problem (P), we shall define a vector-valued Lagrangian $L : Q_0 \times \mathcal{U} \to R^l$ by

$$L(x,\ U) = f(x) + Ug(x). \tag{3. 1}$$

Now, we shall define a saddle point for the vector-valued Lagrangian $L(x,\ U)$.

DEFINITION 3. 1. A point $(x^*,\ U^*) \in Q_0 \times \mathcal{U}$ is said to be a saddle point of the vector-valued Lagrangian $L(x,\ U)$ with respect Q_0 and $\mathcal{U}$ if it holds that

$$L(x^*,\ U) \leqq L(x^*,\ U^*) \leqq L(x,\ U^*),\ \ {}^{\forall}x \in Q_0,\ \ {}^{\forall}U \in \mathcal{U}. \tag{3. 2}$$

(3. 2) may be written as follows:

$$x^* \quad \text{minimizes } L(x,\ U^*) \text{ over } Q_0,$$

$$U^* \quad \text{maximizes } L(x^*,\ U) \text{ over } \mathcal{U}.$$

So, we may extend the concept of the saddle point as follows.

DEFINITION 3. 2. A point $(x^*,\ U^*) \in Q_0 \times \mathcal{U}$ is said to be a Pareto saddle point of the vector-valued Lagrangian $L(x,\ U)$ with respect to Q_0 and $\mathcal{U}$ if it holds that

$$L(x,\ U^*) \nleq L(x^*,\ U^*) \nleq L(x^*,\ U),\ \ {}^{\forall}x \in Q_0,\ \ {}^{\forall}U \in \mathcal{U}, \tag{3. 3}$$

where (3. 3) means that there exists no $(x,\ U) \in Q_0 \times \mathcal{U}$ such that

(i) $L(x,\ U^*) \leq L(x^*,\ U^*)$,

or

(ii) $L(x^*,\ U^*) \leq L(x^*,\ U)$.

DEFINITION 3. 3. A point $(x^*,\ U^*) \in Q_0 \times \mathcal{U}$ is said to be a weak

Pareto saddle point of the vector-valued Lagrangian $L(x, U)$ with respect to Q_0 and $\mathcal{U}$ if it holds that

$$L(x, U^*) \nless L(x^*, U^*) \nless L(x^*, U), \quad {}^{\forall}x \in Q_0, \quad {}^{\forall}U \in \mathcal{U}. \tag{3. 4}$$

The following lemma gives necessary and sufficient conditions for $(x^*, U^*) \in Q_0 \times \mathcal{U}$ to be a saddle point of the vector-valued Lagrangian $L(x, U)$ with respect to Q_0 and $\mathcal{U}$.

LEMMA 3. 1. *A point $(x^*, U^*) \in Q_0 \times \mathcal{U}$ is a saddle point of the vector-valued Lagrangian $L(x, U)$ with respect to Q_0 and $\mathcal{U}$ if and only if the following conditions hold*:

$$g(x^*) \leqq 0, \tag{3. 5}$$

$$U^* g(x^*) = 0, \tag{3. 6}$$

$$f(x^*) \leqq f(x) + U^* g(x), \quad {}^{\forall}x \in Q_0. \tag{3. 7}$$

PROOF. Let $(x^*, U^*) \in Q_0 \times \mathcal{U}$ be a saddle point of the vector-valued Lagrangian $L(x, U)$ with respect to Q_0 and $\mathcal{U}$. Then from the first inequality of (3. 2), we get

$$Ug(x^*) \leqq U^* g(x^*)$$

for all $U \in \mathcal{U}$. Since U can be taken large arbitrarily, it follows that

$$g(x^*) \leqq 0.$$

By letting $U = 0$ in (3. 2), it follows that

$$U^* g(x^*) \geqq 0.$$

Also, since $U^* \geqq 0$ and $g(x^*) \leqq 0$, it follows that

$$U^* g(x^*) = 0.$$

Hence, for all $x \in Q_0$, it follows that

$$f(x^*) \leqq f(x) + U^* g(x),$$

which shows that (3. 7) holds.

Conversely, suppose that (3. 5), (3. 6), and (3. 7) hold. Then, from (3. 5) and (3. 6), for all $U \in \mathcal{U}$, it follows that

$$f(x^*) + Ug(x^*) \leqq f(x^*) + U^* g(x^*).$$

Also, from (3. 6) and (3. 7), for all $x \in Q_0$, it follows that

$$f(x^*) + U^*g(x^*) \leqq f(x) + U^*g(x).$$

This completes the proof. Q. E. D.

Next we shall give necessary and sufficient conditions for $(x^*, U^*) \in Q_0 \times \mathcal{U}$ to be a Pareto saddle point of the vector-valued Lagrangian $L(x, U)$ with respect to Q_0 and $\mathcal{U}$.

LEMMA 3. 2 *A point $(x^*, U^*) \in Q_0 \times \mathcal{U}$ is a Pareto saddle point of the vector-valued Lagrangian $L(x, U)$ with respect to Q_0 and $\mathcal{U}$ if and only if the following conditions hold:*

$$g(x^*) \leqq 0, \tag{3. 8}$$

$$U^*g(x^*) = 0, \tag{3. 9}$$

$$f(x) + U^*g(x) \nleq f(x^*), \quad \forall x \in Q_0. \tag{3. 10}$$

PROOF. Let $(x^*, U^*) \in Q_0 \times \mathcal{U}$ be a Pareto saddle point of the vector-valued Lagrangian $L(x, U)$ with respect to Q_0 and $\mathcal{U}$. Then by Definition 3. 2, it follows that

$$U^*g(x^*) \nleq Ug\ (x^*) \qquad \text{for all } U \in \mathcal{U}.$$

Since U can be taken large arbitrarily, it follows that

$$g(x^*) \leqq 0.$$

By letting $U=0$ in (3. 3), it follows that

$$U^*g(x^*) \nleq 0.$$

Since $U^* \geqq 0$ and $g(x^*) \leqq 0$, it follows that

$$U^*g(x^*) = 0.$$

Hence, from (3. 3), for all $x \in Q_0$, we get

$$f(x) + U^*g(x) \nleq f(x^*).$$

Conversely, suppose that (3. 8), (3. 9), and (3. 10) hold. Then, from (3. 8) and (3. 9), for all $U \in \mathcal{U}$, it follows that

$$f(x^*) + Ug(x^*) \leqq f(x^*) + U^*g(x^*).$$

Also, from (3. 9) and (3. 10), for all $x \in Q_0$, it follows that

$$f(x) + U^*g(x) \nleq f(x^*) + U^*g(x^*).$$

This completes the proof. Q. E. D.

The following theorem gives sufficient conditions for optimality to the problem (P).

THEOREM 3. 1. *If a point* $(x^*, U^*) \in Q_0 \times \mathcal{U}$ *is a saddle point of the vector-valued Lagrangian* $L(x, U)$ *with respect to* Q_0 *and* $\mathcal{U}$, *then* $x^* \in Q_0$ *is an optimal solution to the problem* (P).

PROOF. By Lemma 3. 1, it suffices to show that

$$f(x^*) \leqq f(x)$$

for all $x \in Q$. Since $U^* \geqq 0$ and $g(x) \leqq 0$, for any $x \in Q$, it follows that

$$f(x^*) \leqq f(x) + U^* g(x) \leqq f(x) \qquad \text{for all } x \in Q,$$

which shows that $x^* \in Q_0$ is an optimal solution to the problem (P).

Q. E. D.

The following theorem gives sufficient conditions for Pareto optimality to the problem (P).

THEOREM 3. 2. *If a point* $(x^*, U^*) \in Q_0 \times \mathcal{U}$ *is a Pareto optimal saddle point of the vector-valued Lagrangian* $L(x, U)$ *with respect to* Q_0 *and* $\mathcal{U}$, *then* $x^* \in Q_0$ *is a Pareto optimal solution to the problem* (P).

PROOF. Let $(x^*, U^*) \in Q_0 \times \mathcal{U}$ be a Pareto saddle point of the vector-valued Lagrangian $L(x, U)$ with respect to Q_0 and $\mathcal{U}$. By Lemma 3. 2, it suffices to show that there exists no $x \in Q$ such that

$$f(x) \leq f(x^*).$$

Suppose that there exists an $\bar{x} \in Q$ such that

$$f(\bar{x}) \leq f(x^*).$$

Since $U^* \geqq 0$ and $g(\bar{x}) \leqq 0$, it follows that

$$f(\bar{x}) + U^* g(\bar{x}) \leq f(x^*).$$

This contradicts (3. 3). This completes the proof. Q. E. D.

The following theorem gives sufficient conditions for weak Pareto optimality to the problem (P).

THEOREM 3. 3. *Let a point* $(x^*, U^*) \in Q_0 \times \mathcal{U}$ *be a weak Pareto saddle point of the vector-valued Lagrangian* $L(x, U)$ *with respect to* Q_0

and $\mathcal{U}$. *If* $U^*g(x^*)=0$, *then* $x^* \in Q_0$ *is a weak Pareto optimal solution to the problem* (P).

PROOF. Let $(x^*, U^*) \in Q_0 \times \mathcal{U}$ be a weak Pareto saddle point of the vector-valued Lagrangian $L(x, U)$ with respect to Q_0 and $\mathcal{U}$. Then by Definion 3. 3, we get

$$f(x) + U^*g(x) \not< f(x^*)$$

for all $x \in Q_0$. Suppose that there exists an $\bar{x} \in Q$ such that

$$f(\bar{x}) < f(x^*).$$

Since $U^* \geqq 0$ and $g(\bar{x}) \leqq 0$, it follows that

$$f(\bar{x}) + U^*g(\bar{x}) < f(x^*) = f(x^*) + U^*g(x^*).$$

This contradicts (3. 4). This completes the proof. Q. E. D.

The following corollary is derived from Theorem 3. 3.

COROLLARY 3. 3. 1. *Let a point* $(x^*, U^*) \in Q_0 \times \mathcal{U}$ *is a weak Pareto saddle point of the vector-valued Lagrangian* $L(x, U)$ *with respect to* Q_0 *and* $\mathcal{U}$. *If* $U^{*T} = (u^*, u^*, \dots, u^*)$, *then* $x^* \in Q_0$ *is a weak Pareto optimal solution to the problem* (P).

PROOF. By Theorem 3. 3, it suffices to prove that

$$U^*g(x^*) = 0.$$

Let $(x^*, U^*) \in Q_0 \times \mathcal{U}$ be a weak Pareto saddle point of the vector-valued Lagrangian $L(x, U)$ with respect to Q_0 and $\mathcal{U}$. Then by Definition 3. 3, we get

$$U^*g(x^*) \not< Ug(x^*)$$

for all $U \in \mathcal{U}$. Since U can be taken large arbitrarily, it follows that

$$g(x^*) \leqq 0.$$

Since $U^* \geqq 0$ and $g(x^*) \leqq 0$, it follows that

$$U^*g(x^*) \leqq 0.$$

By letting $U = 0$ in (3. 4), it follows that

$$U^*g(x^*) \not< 0,$$

which implies $u^* \cdot g(x^*) = 0$. Hence, it follows that

$$U^* g(x^*) = 0.$$

This completes the proof. Q. E. D.

4. Necessary Conditions for Optimality

In this section, we shall give necessary and sufficient conditions for optimality, Pareto optimality and weak Pareto optimality in terms of the corresponding saddle points of the vector-valued Lagrangian.

First, we shall give necessary and sufficient conditions for optimality to the problem (P).

THEOREM 4. 1. *Suppose that all the hypotheses of Lemma* 2. 1 *hold. Then* $x^* \in Q_0$ *is an optimal solution to the problem* (P) *if and only if there exists a matrix* $U^* \in \mathcal{U}$ *such that* $(x^*, U^*) \in Q_0 \times \mathcal{U}$ *is a saddle point of the vector-valued Lagrangian* $L(x, U)$ *with respect to* Q_0 *and* $\mathcal{U}$.

PROOF. Since the 'if' part has been proved in Theorem 3. 1, it suffices to prove the 'only if' part. Let $x^* \in Q_0$ be an optimal solution to the problem (P). Then x^* minimizes $f_i(x)$ over Q for all i. Hence, for each i $(i=1, 2,\dots, l)$, from the Kuhn-Tucker saddle point theorem, there exists a vector $u^i \in R^m$ such that

$$u^i \geqq 0, \tag{4. 1}$$

$$f_i(x^*) + u \cdot g(x^*) \leqq f_i(x^*) + u^i \cdot g(x^*) \leqq f_i(x) + u^i \cdot g(x), \quad \forall x \in Q_0, \ \forall u \in R^m_+; \tag{4. 2}$$

$$g(x^*) \leqq 0, \tag{4. 3}$$

$$u^i \cdot g(x^*) = 0. \tag{4. 4}$$

Define U^* by

$$U^{*T} = (u^1, u^2, \dots, u^l).$$

Then we get $U^* \in \mathcal{U}$ and

$$L(x^*, U) \leqq L(x^*, U^*) \leqq L(x, U^*)$$

for all $x \in Q_0$ and $U \in \mathcal{U}$. This complete the proof. Q. E. D.

The following theorem gives necessary conditions for weak Pareto optimality to the problem (P).

THEOREM 4. 2. *Suppose that all the hypotheses of Lemma* 2. 1 *hold. Then* $x^* \in Q_0$ *is a weak Pareto optimal solution to the problem* (P) *if and only if there exists a vector* $u^* \in R^m_+$ *such that* $(x^*, U^*) \in Q_0 \times \mathcal{U}$ *is a weak Pareto saddle point of the vector-valued Lagrangian* $L(x, U)$ *with respect to* Q_0 *and* $\mathcal{U}$, *where* $U^{*T} = (u^*, u^*, \ldots, u^*)$.

PROOF. Since the 'if' part has been proved in Corollary 3. 3. 1, it suffices to prove the 'only if' part. Let $x^* \in Q_0$ be a weak Pareto optimal solution to the problem (P). Then by Lemma 2. 1, there exists vectors $v^* \in R^l$ and $u^* \in R^m$ such that

$$v^* \geq 0, \quad u^* \geqq 0, \tag{4. 5}$$

$$v^* \cdot f(x^*) + u \cdot g(x^*) \leqq v^* \cdot f(x^*) + u^* \cdot g(x^*) \leqq v^* \cdot f(x) + u^* \cdot g(x), \quad \forall x \in Q_0, \forall U \in \mathcal{U}, \tag{4. 6}$$

$$g(x^*) \leqq 0, \tag{4. 7}$$

$$u^* \cdot g(x^*) = 0. \tag{4. 8}$$

Without any loss of generality, we may assume that $\sum_{i=1}^{l} v_i^* = 1$. Define U^* by

$$U^{*T} = (u^*, u^*, \ldots, u^*).$$

Then we get $U^* \in \mathcal{U}$, and

$$U^* g(x^*) = 0.$$

Hence, we get

$$f(x^*) + Ug(x^*) \leqq f(x^*) + U^* g(x^*)$$

for all $U \in \mathcal{U}$. This shows that the first inequality of (3. 2) holds.

Next we shall show that there exists no $x \in Q_0$ such that

$$L(x, U^*) < L(x^*, U^*).$$

Suppose that there exists an $\bar{x} \in Q_0$ such that

$$L(\bar{x}, U^*) < L(x^*, U^*).$$

Then for $v^* \geq 0$ in (4. 5), we get

$$v^* \cdot f(\bar{x}) + u^* \cdot g(\bar{x}) < v^* \cdot f(x^*) + u^* \cdot g(x^*).$$

This contradicts (4. 6). This completes the proof. Q. E. D.

The following theorem gives necessary and sufficient conditions for Pareto optimality to the problem (P).

THEOREM 4. 3. *Suppose that all the hypotheses of Lemma* 2. 2 *hold. Then* $x^* \in Q_0$ *is a Pareto optimal solution to the problem* (P) *if and only if there exists a matrix* $U^* \in \mathcal{U}$ *such that* $(x^*, U^*) \in Q_0 \times \mathcal{U}$ *is a Pareto saddle point of the vector-valued Lagrangian* $L(x^*, U^*)$ *with respect to* Q_0 *and* $\mathcal{U}$.

PROOF. Since the 'if' part has been proved in Theorem 3. 2, it suffices to prove the 'only if' part. Let $x^* \in Q_0$ be a Pareto optimal solution to the problem (P). Then by Lemma 2. 2, there exist vectors $v^* \in R^l$ and $u^* \in R^m$ such that

$$v^* > 0, \ u^* \geqq 0, \tag{4. 9}$$

$$v^* \cdot f(x^*) + u \cdot g(x^*) \leqq v^* \cdot f(x^*) + u^* \cdot g(x^*) \leqq v^* \cdot f(x) + u^* \cdot g(x),$$
$$\forall x \in Q_0, \ \forall u \in R_+^m \tag{4. 10}$$

$$g(x^*) \leqq 0, \tag{4. 11}$$

$$u^* \cdot g(x^*) = 0. \tag{4. 12}$$

Without any loss of generality, we may assume that $\sum_{i=1}^{l} v_i^* = 1$. Define U^* by

$$U^{*T} = (u^*, \ u^*, \ldots, \ u^*).$$

Then we get $U^* \in \mathcal{U}$, and

$$U^* g(x^*) = 0.$$

Hence, we get

$$f(x^*) + U g(x^*) \leqq f(x^*) + U^* g(x^*)$$

for all $U \in \mathcal{U}$. This shows that the first inequality of (3. 2) holds.

Next we shall show that there exists no $x \in Q_0$ such that

$$L(x, U^*) \leq L(x^*, U^*).$$

Suppose that there exists an $\bar{x} \in Q_0$ such that

$$L(\bar{x}, U^*) \leq L(x^*, U^*).$$

Then for $v^* > 0$ in (4.9), we get

$$v^* \cdot f(\bar{x}) + u^* \cdot g(\bar{x}) < v^* \cdot f(x^*) + u^* \cdot g(x^*).$$

This contradicts (4.10). This completes the proof. Q. E. D.

From Theorem 4. 3, the following corollary is derived easily.

COROLLARY 4. 3. 1 *Suppose that all the hypotheses of Lemma* 2. 2 *hold. Then* $x^* \in Q_0$ *is a Pareto optimal solution to the problem* (P) *if and only if there exists a vector* $u^* \in R^m_+$ *such that* $(x^*, U^*) \in Q_0 \times \mathcal{U}$ *is a Pareto saddle point of the vector-valued Lagrangian* $L(x, U)$ *with respect to* Q_0 *and* $\mathcal{U}$, *where* $U^{*T} = (u^*, u^*, \ldots, u^*)$.

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